

## Power Series

Any series of the form

$\sum_{n=0}^{\infty} a_n (z - z_0)^n$  is called power series

## Pointwise Convergence

A sequence of functions  $\{f_n(z)\}$  converges pointwise to  $f(z)$  on Region  $R$  if for each  $z$ , given  $\epsilon > 0$ ,  $\exists$  a natural no  $n_1$  such that

$$|f_n(z) - f(z)| < \epsilon \quad \forall n \geq n_1$$

$\{f_n(z)\}$  converges uniformly in  $R$  if for every  $\epsilon > 0$ , we can find  $n_1$  depending on  $\epsilon$  only such that

$$|f_n(z) - f(z)| < \epsilon \quad \forall n \geq n_1, \quad \forall z$$

Here  $n_1$  doesn't depend upon  $z$

e.g.  $f_n(z) = \frac{1}{n^2} \quad n = 1, 2, \dots$

in annular region

converges uniformly to 0  $1 \leq |z| \leq 2$   
given  $\epsilon > 0$

$$|f_n(z) - f(z)| = \left| \frac{1}{n^2} - 0 \right| = \frac{1}{n^2}$$

$$\therefore \frac{1}{n^2} \leq \frac{1}{n}$$

$$\Rightarrow \frac{1}{n^2} \leq \frac{1}{n}$$

$$\forall z \quad 1 \leq |z| \leq 2$$

$$\text{Let } \frac{1}{n} < \epsilon \Rightarrow n > \frac{1}{\epsilon}$$

Thus for any  $\epsilon > 0$ , we can find  $n_1$  traced by  $\frac{1}{\epsilon}$  such that

$$|f_n(z) - 0| < \epsilon$$

$$\forall n > n_1$$

$$\forall z, \phi \in \mathbb{R}, |\phi| \leq 2$$

Note Consider a series

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \dots + z_n + z_{n+1} + z_{n+2} + \dots$$

$$= z_1 + z_2 + z_3 + \dots + z_n + R_n$$

where  $R_n$  is remainder after  $n$  terms.

$$= S_n + R_n$$

$S_n$  is  $n$ th partial sum

Now if the sum of the series is a given no  $S$  i.e. series converges to  $S$

Then  $\sum_{n=1}^{\infty} z_n = S$  and we have

$$S = S_n + R_n$$

$$S - S_n = R_n$$

$$|S_n - S| = |R_n|$$

$$\therefore S_n \rightarrow S \text{ as } n \rightarrow \infty$$

$$\therefore R_n \rightarrow 0 \text{ as } n \rightarrow \infty$$

Conversely if  $R_n \rightarrow 0$ , then  $S_n \rightarrow S$

Thus series converges to a number  $S$  iff sequence of remainders  $\{R_n\}$  tends to zero

### 3 Region of Convergence and Radius of Convergence of power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Let  $T_n = a_n (z - z_0)^n$

$$T_{n+1} = a_{n+1} (z - z_0)^{n+1}$$

$$\left| \frac{T_{n+1}}{T_n} \right| = \left| \frac{a_{n+1}}{a_n} \right| |z - z_0|$$

$$\lim_{n \rightarrow \infty} \left| \frac{T_{n+1}}{T_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| |z - z_0|$$

Now series will converge

If  $\lim_{n \rightarrow \infty} \left| \frac{T_{n+1}}{T_n} \right| < 1$

If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| |z - z_0| < 1$

If  $|z - z_0| < \frac{1}{\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|} = R$

provided  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$  exist

Here  $R = \frac{1}{\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|}$  is radius of convergence

and  $|z - z_0| < R$  is region of convergence

Again  $|T_n|^{1/n} = |a_n|^{1/n} |z - z_0|$

$$\lim_{n \rightarrow \infty} |T_n|^{1/n} = \lim_{n \rightarrow \infty} |a_n|^{1/n} |z - z_0|$$

Series will converge

If  $\lim_{n \rightarrow \infty} |T_n|^{1/n} < 1$

If  $\lim_{n \rightarrow \infty} |a_n|^{1/n} |z - z_0| < 1$

$$\Rightarrow |z-z_0| < \frac{1}{\lim_{n \rightarrow \infty} |a_n|^{1/n}} = R$$

$$\text{Thus } R = \frac{1}{\lim_{n \rightarrow \infty} |a_n|^{1/n}}$$

Note (1) at  $z = z_0$  series  $= \sum 0$  which is cgt  
Thus at least series converges at  $z = z_0$  and in this case radius of convergence is  $\geq 0$

(2) If  $R = \infty$ , then series converges in whole complex plane

Theorem (Abel's Theorem)

If the power series  $\sum_{n=0}^{\infty} a_n (z-a)^n$  is cgt absolutely at  $z_0$ , then it converges absolutely and uniformly in set  $\{z \in \mathbb{C} : |z-a| \leq |z_0-a|\}$  i.e. for all  $z$  for which  $|z-a| \leq |z_0-a|$

Proof  $\because |z_0-a|$  is constant

$$\therefore \text{let } |a_n| |z_0-a| = M_n$$

$$\text{Now } |z-a| \leq |z_0-a|$$

$$\Rightarrow |a_n| |z-a| \leq |a_n| |z_0-a| = M_n$$

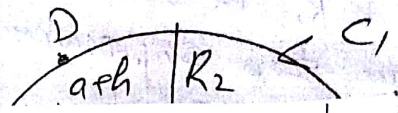
$\Rightarrow$  New series  $\sum M_n = \sum_{n=0}^{\infty} |a_n| |z_0-a|$  is cgt by hypothesis

Therefore by Weierstrass M-Test

$$\sum |a_n| |z-a| \text{ is cgt}$$

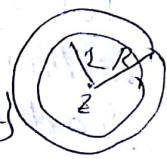
Weierstrass Test

If  $|f_n(z)| \leq M_n$  in region  $R$ , where  $M_n$  of  $\sum M_n$  converges, i.e.  $\sum_{n=0}^{\infty} M_n$  converges, then  $\sum_{n=0}^{\infty} f_n(z)$  converges uniformly



5

Theorem Let  $R$  be radius of convergence of power series  $\sum a_n (z-z_0)^n$ . Then for any  $R$  such that  $0 < R < R$ , the power series converges absolutely and uniformly on closed disc  $\bar{B}(z_0, R) = \{z : |z-z_0| \leq R\}$



Proof For any put  $z_1$  such that

$$|z_1 - z_0| \leq R < R$$

the series converges absolutely

$$\text{Also let } M_n = |a_n| |z_1 - z_0|^n \leq |a_n| R^n$$

Then for any  $z \in \bar{B}(z_0, R)$

$$|a_n| |z - z_0|^n \leq M_n \quad \forall n$$

Then  $\sum M_n$  converges  $\forall z \in \bar{B}(z_0, R)$

$\Rightarrow \sum |a_n| |z - z_0|^n$  converges uniformly

on  $\bar{B}(z_0, R)$

Corollary A power series  $\sum a_n (z-z_0)^n$  defines a continuous function at every  $z$  in its domain  $\{z : |z-z_0| < R\}$  of convergence

Proof If  $z_1$  is such that  $|z_1 - z_0| < R$ , then we can find  $R' > 0$  such that  $|z_1 - z_0| < R' < R$ . Then power series converges uniformly on  $\bar{B}(z_0, R')$  and hence defines a continuous function on it. In particular it is continuous at  $z_1$

Note A polynomial  $P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$  with  $a_n \neq 0$  is a power series at  $z = z_0$  with coefficients  $a_k$   $\forall k \geq 0$

Also power series  $P(z) = \sum a_n z^n$  can be written as

$$P(z) = (a_0 + a_1 z + \dots + a_{n-1} z^{n-1}) + a_n z^n + \dots$$

$$= P_n(z) + R_n(z) \quad \text{where } R_n(z) = \sum_{j=n+1}^{\infty} a_j z^j$$

is the  $n$ th remainder  
 $P_n(z)$  partial sum of  $(a_n)$  terms

$$|f_n(z) - f(z)| \leq \frac{1}{n} |z|$$

$$\therefore |z| \geq 1 \quad \therefore \frac{1}{|z|} \leq 1$$

$$\Rightarrow \frac{1}{n|z|} \leq \frac{1}{n} \quad \forall z, 1 \leq |z| \leq 2$$

# Taylor's Theorem or Expansion

Let  $f$  be analytic at all points within a circle  $C$  centred at  $(a)$  and radius  $r$ , then for any point  $z$  within  $C$

we have

$$f(z) = f(a) + (z-a)f'(a) + \dots + \frac{(z-a)^n}{n!} f^{(n)}(a)$$

$$= \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^{(n)}(a)$$

Proof Let  $C$  be circle  $|z-a| = r$  and let  $z = a+h$  be an interior point of  $C$ . Then By Cauchy integral formula.

$$f(a+h) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a-h} dz \rightarrow (1)$$



$$\frac{1}{z-a-h} = \frac{1}{z-a} + \frac{h}{(z-a)^2} + \dots + \frac{h^n}{(z-a)^{n+1}} + \frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)}$$

$$(z-a-h) \left[ \frac{1}{z-a} + \frac{h}{(z-a)^2} + \dots + \frac{h^n}{(z-a)^{n+1}} \right] = \frac{1}{1 - \frac{h}{z-a}} = \frac{1}{1 - \frac{h}{z-a}}$$

① becomes

$$f(a+h) = \frac{1}{2\pi i} \int_C \left[ \frac{1}{z-a} + \frac{h}{(z-a)^2} + \dots + \frac{h^n}{(z-a)^{n+1}} + \frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)} \right] f(z) dz$$

Using formula

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz \rightarrow (2)$$

$$\frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} f^{(n)}(a) + R_n \rightarrow (3)$$

$$\text{where } R_n = \frac{h^{n+1}}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}(z-a-h)} dz \rightarrow (4)$$

$$|f_n(z) - f(z)|$$

Series ③ will represent  $f(z)$  in  $C$  if  $\sum_{n=0}^{\infty} R_n > 0$

from ④

$$R_n = \frac{f^{(n+1)}(a)}{(n+1)!} \int_C \frac{f(z) dz}{(z-a)^{n+1} (z-a-h)}$$

$$|R_n| = \left| \frac{f^{(n+1)}(a)}{(n+1)!} \int_C \frac{f(z) dz}{(z-a)^{n+1} (z-a-h)} \right|$$

$$\leq \frac{|f^{(n+1)}(a)|}{(n+1)!} \int_C \frac{|f(z)| |dz|}{|z-a-h|^{n+1}}$$

$$\leq \frac{|f^{(n+1)}(a)|}{(n+1)!} \int_C \frac{|f(z)| |dz|}{|z-a-h|^{n+1}} \quad |z-a| = r$$

$$\leq \frac{1}{(n+1)!} \frac{|f^{(n+1)}(a)|}{r^{n+1}} \int_C \frac{|f(z)| |dz|}{|z-a-h|} \rightarrow ①$$

$\therefore a+h$  lies within  $C$ .  
 $\therefore |a+h| < r \Rightarrow \left(\frac{|a+h|}{r}\right)^{n+1} < 1$

$$< \frac{1}{(n+1)!} \int_C \frac{|f(z)| |dz|}{|z-a-h|}$$

if  $|f(z)| \leq M$  on  $C$ , then  $a+h$  lies within  $C$ , therefore  $\frac{f(z)}{z-a-h}$  is analytic on  $C$  and is bounded.

$$\Rightarrow \frac{1}{|z-(a+h)|} \frac{|f(z)|}{|z-a-h|} \leq M$$

for ① by ML inequality

$$|R_n| \leq \frac{|f^{(n+1)}(a)|}{(n+1)!} \frac{M \cdot 2\pi r}{r^{n+1}} < \frac{M |f^{(n+1)}(a)|}{r^n} \left(\frac{r}{r}\right)^n \rightarrow 0$$

Hence proved.

## 8 Laurent's Series (or Expansion)

Suppose that a function  $f$  is analytic

throughout an annular region

$R_1 < |z-a| < R_2$  centred at  $a$  and

let  $C$  denote any evenly oriented <sup>bounded by concentric circles</sup> simple closed contour around  $z=a$   <sub>$C_1: |z-a|=R_1$   
 $C_2: |z-a|=R_2$</sub>

lying in the annular domain. Then at each point  $z$  in the domain  $f(z)$  has

the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} \rightarrow \textcircled{1}$$

where  $a_n = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{(z-a)^{n+1}} \quad (n=0, 1, 2, \dots)$

$$= \frac{1}{2\pi i} \int_{C_1} \frac{f(z) dz}{(z-a)^{n+1}} \quad (n=0, 1, 2, \dots)$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{(z-a)^{-n+1}} = \frac{1}{2\pi i} \int_{C_2} \frac{f(z) dz}{(z-a)^{-n+1}}$$

Expression  $\textcircled{1}$  is often

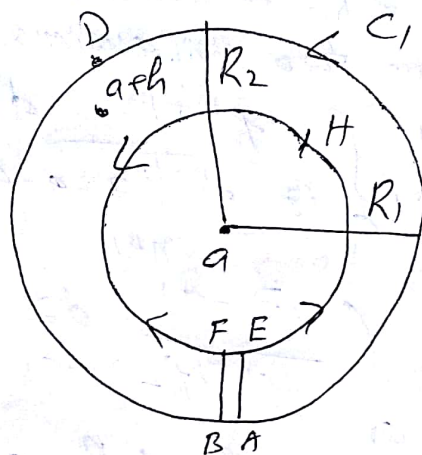
written as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \quad (R_1 < |z-a| < R_2)$$

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{(z-z_0)^{n+1}} \quad (n=0, \pm 1, \pm 2, \dots)$$

9

Proof



Let  $C_1$  &  $C_2$  be two concentric circles with centre at  $(a)$  with radii  $R_1$  &  $R_2$  respectively. Let  $z = a+h$  be any point in the annular region.  $f(z)$  is analytic on  $C_1$  &  $C_2$  and in the annular region and by Cauchy integral formula for doubly connected region, we have closed contour  $A D B F H E A$  and

$$f(z) = f(a+h) = \frac{1}{2\pi i} \int_{ADB F H E A} \frac{f(z) dz}{z - (a+h)}$$

$$= \frac{1}{2\pi i} \int_{C_1} \frac{f(z) dz}{z - (a+h)} - \frac{1}{2\pi i} \int_{C_2} \frac{f(z) dz}{z - a - h} \quad \rightarrow (A)$$

Now  $\frac{1}{z - (a+h)} = \frac{1}{z - a - h} = \frac{1}{z - a} + \frac{h}{(z - a)^2} + \frac{h^2}{(z - a)^3} + \dots + \frac{h^{n+1}}{(z - a)^{n+1}} + \dots$

and  $\frac{1}{h - (z - a)} = \frac{1}{h} + \frac{(z - a)}{h^2} + \frac{(z - a)^2}{h^3} + \dots + \frac{(z - a)^n}{h^{n+1}} + \frac{(z - a)^{n+1}}{h^{n+1}(z + a + h)} + \dots$

using (B) & (C) we get

$$f(a+h) = \frac{1}{2\pi i} \int_{C_1} \left[ \frac{1}{z - a} + \frac{h}{(z - a)^2} + \dots + \frac{h^n}{(z - a)^{n+1}} + \frac{h^{n+1}}{(z - a)^{n+1}(z - a - h)} + \dots \right] dz$$

$$+ \frac{1}{2\pi i} \int_{C_2} \left[ \frac{1}{h} + \frac{(z - a)}{h^2} + \frac{(z - a)^2}{h^3} + \dots + \frac{(z - a)^n}{h^{n+1}} + \frac{(z - a)^{n+1}}{h^{n+1}(z + a + h)} + \dots \right] dz$$

Let  $R_n$  &  $R_n'$  be remainders after  $n$  terms in above two expansions.

$$R_n = \frac{1}{2\pi i} \int_{C_1} \frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)} dz$$

$$R_n' = \frac{1}{2\pi i} \int_{C_2} \frac{(z-a)^{n+1}}{h^{n+1}(h-z+a)} dz$$

We show that  $R_n$  &  $R_n' \rightarrow 0$  as  $n \rightarrow \infty$

$$R_n = \frac{1}{2\pi i} \int_{C_1} \frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)} dz$$

$$|R_n| = \frac{1}{2\pi} \left| \int_{C_1} \frac{h^{n+1}}{(z-a)^{n+1}} \cdot \frac{f(z)}{z-a-h} dz \right|$$

$$\leq \frac{1}{2\pi} \int_{C_1} \left| \frac{f(z)}{z-a-h} \right| \left| \frac{h^{n+1}}{(z-a)^{n+1}} \right| |dz|$$

$$\text{Now } \left| \frac{f(z)}{z-a-h} \right| < M \quad \& \quad \int_C |dz| = 2\pi R_1$$

$$\text{and } |z-a| \geq R_1 \quad \therefore \frac{|h|^{n+1}}{R_1^{n+1}} = MR \left( \frac{|h|}{R_1} \right)^{n+1}$$

$$|R_n| \leq \frac{1}{2\pi} M \cdot 2\pi R \cdot \frac{|h|^{n+1}}{R_1^{n+1}} = MR \left( \frac{|h|}{R_1} \right)^{n+1}$$

$$\text{as } n \rightarrow \infty \quad \frac{|h|}{R_1} \rightarrow 0 \quad \left( \frac{|h|}{R_1} < 1 \right)$$

$$\therefore |R_n| \rightarrow 0$$

11

$$|R_n'| = \left| \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{h-z+a} \frac{(z-a)^{n+1}}{h^{n+1}} dz \right|$$

$$\leq \frac{1}{2\pi} \int_{C_2} |dz| \left| \frac{f(z)}{h-z+a} \right| \left| \frac{(z-a)^{n+1}}{h^{n+1}} \right|$$

$$\leq \frac{1}{2\pi} M \cdot 2\pi R_2 \cdot \frac{R_2^{n+1}}{|h|^{n+1}} = MR_2 \left( \frac{R_2}{|h|} \right)^{n+1}$$

→ so as  $n \rightarrow \infty$ ,  
 $\therefore \frac{R_2}{|h|} < 1$